



General recurrence and ladder relations of hypergeometric-type functions[☆]

A. Zarzo^{a, d}, R.J. Yáñez^{b, d}, J.S. Dehesa^{c, d, *}

^a*Departamento de Matemática Aplicada, E.T.S. Ingenieros Industriales, Universidad Politécnica de Madrid, Spain*

^b*Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Granada, E-18071 Granada, Spain*

^c*Departamento de Física Moderna, Facultad de Ciencias, Universidad de Granada, E-18071 Granada, Spain*

^d*Instituto Carlos I de Física Teórica y Computacional, Facultad de Ciencias, Universidad de Granada, Spain*

Abstract

A method for the explicit construction of general linear sum rules involving hypergeometric-type functions and their derivatives of any order is developed. This method only requires the knowledge of the coefficients of the differential equation that they satisfy, namely the hypergeometric-type differential equation. Special attention is paid to the differential-recurrence or ladder relations and to the fundamental three-term recurrence formulas. Most recurrence and ladder relations published in the literature for numerous special functions including the classical orthogonal polynomials, are instances of these sum rules. Moreover, an extension of the method to the generalized hypergeometric-type functions is also described, allowing us to obtain explicit ladder operators for the radial wave functions of multidimensional hydrogen-like atoms, where the varying parameter is the dimensionality.

© 2006 Elsevier B.V. All rights reserved.

MSC: 34A25; 33C05; 33C45

Keywords: Second-order differential equations; Special functions; Hypergeometric-type functions; Classical orthogonal polynomials; Recurrence formulas; Ladder operators; Sum rules

1. Introduction

The study of linear and nonlinear N -term ($N \geq 3$) relations among the special functions of mathematical physics and their derivatives of any order is a very relevant mathematical problem from both fundamental [3,8,12–14,19,20,23,26,28,41,43,44] and applied [2,4,9,11,29,31,32,46] standpoints. If $\{y_\alpha(z)\}_{\alpha \in A}$ denotes a specific one-parameter family of special functions, being α the parameter running over a certain set of indices A , a wide and important subset of such algebraic properties can be described by the general sum rule,

$$\sum_{i=0}^{N-1} A_{\alpha_i}(z) \frac{d^{k_i}}{dz^{k_i}} [y_{\alpha_i}(z)] = 0 \quad (N \geq 3, \alpha_i \in A, i = 0, 1, \dots, N-1), \quad (1)$$

[☆] This paper has been written to honor Nico Temme on occasion of his 65th birthday.

* Corresponding author. Departamento de Física Moderna, Facultad de Ciencias, Universidad de Granada, E-18071 Granada, Spain.

E-mail address: dehesa@ugr.es (J.S. Dehesa).

linearly linking $N \geq 3$ different functions and/or their derivatives of arbitrary order. Some interesting particular instances of Eq. (1) are the fundamental three-term recurrence formula ($N = 3$, $k_i = 0$, $\alpha_i = \alpha + i$) and the ladder relations (also called structure relations [3,12] in the theory of orthogonal polynomials), which allows us to find ladder operators (i.e., first-order differential operators shifting the index α by ± 1 , i.e., $N = 3$, $k_0 = 1$, $k_1 = k_2 = 0$, $\alpha_0 = \alpha_1 = \alpha$, $\alpha_2 = \alpha \pm 1$ in Eq. (1)).

The study of these sum rules include the two following problems: to find existence conditions of these relationships and to compute explicitly the coefficients $A_{\alpha_i}(z)$ ($i = 0, 1, \dots, N - 1$) which completely characterize the corresponding algebraic and/or differential properties as given by Eq. (1). For solving these problems some general methods have been designed in the literature which use as starting point some specific property of the involved special functions. Among them, let us mention the early works of Inoui [26] and Truesdell [41], and those of Barik [8] and Hansen [23] where the starting point is a recurrence formula that the functions must satisfy as well as generating function technique developed by Dattoli et al. [13–17]. More specific is the method of Marcellán et al. [28], which provides an unified way of obtaining the three-term recurrence and structure relations for the classical orthogonal polynomials starting from the Pearson-type equation satisfied by the corresponding “orthogonalizing” weight function. Some sum rules could be also obtained when the function admits a representation as a generalized hypergeometric series [1,22,37].

Special attention has been paid to the construction of ladder operators because of the important role they play on solving the fundamental wave equation in relativistic and nonrelativistic quantum mechanics (see [27] for an state-of-the-art of this subject up to the nineties). In this context those operators are usually called “creation” and “annihilation” operators and their significance comes from a Dirac idea [21], which was developed later on, giving rise to the well known “factorization method” [25], for which a second-order differential equation satisfied by the involved functions is required. Also, some alternative approaches for computing recurrences have been devised, not only for solving the wave equation, but also for the computation of matrix elements, mainly referring to hydrogenic and oscillator-like wave functions (see e.g., [9,31,32]).

Let us consider the *generalized hypergeometric-type differential equation* [34], that is the equation

$$u''(z) + \frac{p(z)}{Q(z)} u'(z) + \frac{q(z)}{[Q(z)]^2} u(z) = 0, \quad (2)$$

where $q(z)$ and $Q(z)$ are polynomials of degree at most two, and $p(z)$ is also a polynomial of degree at most one. Solutions of Eq. (2) form an important subset of special functions usually known (in a wide sense) as “special functions of mathematical physics” because of their occurrence in many fundamental problems of applied mathematics and mathematical physics [30,34,5,6,40]. A way to study this equation begins with a set of changes of the independent variable [34] which transforms Eq. (2) into the simpler equation

$$\sigma(z) y''(z) + \tau(z) y'(z) + \lambda y(z) = 0, \quad (3)$$

called by *hypergeometric-type differential equation*, where $\sigma(z)$ and $\tau(z)$ are polynomials of degree not greater than two and one, respectively, and λ is a constant. In fact, any linear second-order differential equation whose singularities are regular and no more than three in number, can be transformed into an equation of this form [35, Theorem 8.1].

In the late eighties Nikiforov and Uvarov proposed a method [34, p. 14], later on developed and extended [20,43–46], to compute algebraic and differential properties of the type (1) for *hypergeometric-type functions*, which are some solutions of Eq. (3). Under appropriate conditions (see [34, p. 14] and [20,43–46]) this method allows us an unified computation of coefficients $A_{\alpha_i}(z)$ in Eq. (1) directly in terms of the generic coefficients $\sigma(z)$ and $\tau(z)$ of the differential equation. This method is based upon the use of an integral representation [34, p. 9] for the solutions of Eq. (3); see also Ref. [43] for a detailed description. Therein, we realize that this method applies to hypergeometric-type equations of the type (3) in which the coefficients $\sigma(z)$ and $\tau(z)$ do not depend on the spectral parameter λ .

The main purpose of this work is to extend the Nikiforov–Uvarov method to the hypergeometric-type differential equation (3) where the coefficients (σ, τ, λ) are parameter dependent, i.e.,

$$\sigma(z; \bar{c}) y'' + \tau(z; \bar{c}) y' + \lambda(\bar{c}) y = 0, \quad (4)$$

where \bar{c} stands for a set of meaningful parameters, being 3 the irreducible number of them, so that the σ and τ coefficients change when the spectral parameter λ does so. This kind of equation are naturally encountered from the generalized

differential equation (2) by means of an appropriate change of the independent variable. Moreover, particular instances of (4) describe [45] fundamental wave equations of numerous classical [38,39] and quantum-mechanical systems in ordinary [2,7,18,29,31,32,36,45] and D -dimensional ($D > 3$) [9,11,33] spaces. So, the extended method provides an *unified approach* to compute N -term recurrence and differential-recurrence relations of the type (1) for a much larger class of hypergeometric-type functions and then to obtain sum rules and ladder operators associated to wave functions of a wide variety of physical systems.

In doing so, we first introduce in Section 2 the required definitions and notations to be used throughout the paper. Section 3 includes an alternative way of constructing the integral representation for the hypergeometric-type functions and their derivatives of any order given in [34, p. 9] which, as already pointed out, will be the cornerstone of our method. Our main result is contained in Section 4, where it is obtained (see Theorem 2 below) a set of sufficient conditions for the sum rules (1) to exist. Then, a constructive method for the explicit computation of the corresponding coefficients in terms of only the polynomials σ and τ in Eq. (4) is developed. As illustration, some general three-term recurrence formulas and some ladder operators are explicitly calculated. In Section 5 it is shown how the algebraic sum rules (1) obtained for the hypergeometric-type functions can be extended to the solutions of the generalized equation (2). This extension is, in fact, a straightforward consequence of the change of variable [34, pp. 1–3] which links both of them.

Finally, in Section 6, our method is applied to some generalized hypergeometric-type functions of physical interest, the radial wave functions of the D -dimensional hydrogen atom [33,42]. We obtain two ladder operators shifting the dimension and so relating in this way the radial wave functions in D dimensions with those in dimensions $D \pm 1$.

2. Basic definitions and notations

Since a linear change of variable can always remove the dependence of the coefficient $\sigma(z; \bar{c})$ in Eq. (4) on the parameters \bar{c} , we will reduce without loss of generality hypergeometric-type equations of the type (4) with $\sigma := \sigma(z)$. On the other hand, following Nikiforov and Uvarov's ideas, we introduce a new parameter $v \in \mathbb{C}$ as a solution of the (at most quadratic) equation

$$\lambda(\bar{c}) + v\tau'(\bar{c}) + \frac{v(v-1)}{2}\sigma'' = 0. \quad (5)$$

Then, we will consider in this paper the hypergeometric-type differential equation

$$\sigma(z)y'' + \tau(z; v, \bar{c})y' + \lambda_v(\bar{c})y = 0. \quad (6)$$

Its symmetrization factor, to be denoted by $w(z; v, \bar{c})$, will play also a relevant role. It is a solution of the Pearson-type differential equation

$$\frac{d}{dz}[\sigma(z) w(z; v, \bar{c})] = \tau(z; v, \bar{c})w(z; v, \bar{c}). \quad (7)$$

Remark 1. The parameter v as defined by Eq. (5) cannot be introduced when $\deg(\sigma) \leq 1$ and $\deg(\tau(z; v, \bar{c})) = 0$. However, if $\deg(\sigma) = 1$ there always exists a change of variable [34,45] which transforms the corresponding hypergeometric-type equation into another of the same form but with $\deg(\sigma) = 1$ and $\deg(\tau(z; v, \bar{c})) = 1$. So that, the definition of the v -parameter, only excludes differential equations with constant coefficients.

Finally, for convenience we define the two following useful functions

$$\tau_a(z; v, \bar{c}) := \tau(z; v, \bar{c}) + a\sigma'(z), \quad w_a(z; v, \bar{c}) := [\sigma(z)]^a w(z; v, \bar{c}), \quad (8)$$

for any complex number $a \in \mathbb{C}$. As a straightforward consequence of the Pearson-type equation (7), these functions satisfy the relation

$$\frac{d}{dz}[\sigma(z) w_a(z; v, \bar{c})] = \tau_a(z; v, \bar{c}) w_a(z; v, \bar{c}), \quad (9)$$

which simplifies to the Pearson-type equation itself when $a=0$. For completeness, let us also mention that the definition of the parameter $v \in \mathbb{C}$ given by Eq. (5) can be simply rewritten as

$$\lambda_v(\bar{c}) + v \tau'_{(v-1)/2}(\bar{c}) = 0, \quad (10)$$

once we take into account the notation given by Eq. (8).

3. An integral representation for solutions of the hypergeometric-type differential equation and their derivatives of any order

As pointed out in the Introduction, the basic tool of our method for computing algebraic and differential properties of the form given by Eq. (1) is an integral representation for the solutions of the hypergeometric-type equation (6), already obtained in [34, Section 3], which give rise to the definition of what it is called hypergeometric-type function. Here, we are going to present an alternative way to the one of [34, Section 3] for constructing this integral representation, which is based on standard techniques coming from the general theory of differential equations (see e.g. [24, p. 186 and p. 438]).

Let us define the differential operators

$$\mathcal{L}_z := \sigma(z) \frac{d^2}{dz^2} + \tau(z; v, \bar{c}) \frac{d}{dz} + \lambda_v(\bar{c}) \mathcal{I}, \quad \mathcal{M}_s := -\sigma(s) \frac{d}{ds} - \tau(z; v, \bar{c}) \mathcal{I}, \quad (11)$$

acting on the variables z and s , respectively, being \mathcal{I} the identity operator. If $\overline{\mathcal{M}}_s$ denotes the formal Lagrange adjoint of \mathcal{M}_s , i.e.,

$$\overline{\mathcal{M}}_s := \sigma(s) \frac{d}{ds} + [\sigma'(s) - \tau_v(z; v, \bar{c})] \mathcal{I}, \quad (12)$$

it turns out that the function $w_v(s; v, \bar{c})$ defined in Eq. (8) satisfies $\overline{\mathcal{M}}_s[w_v(s; v, \bar{c})] = 0$, which is, in fact, Eq. (9). Moreover, the Lagrange identity [24, Section 5.3] reads in this case as follows:

$$v(s) \mathcal{M}_s[u(s)] - u(s) \overline{\mathcal{M}}_s[v(s)] = -\frac{d}{ds} [u(s) \sigma(s) v(s)]. \quad (13)$$

On the other hand, the functions

$$K(z, s) := \frac{1}{w(z; v, \bar{c})(s - z)^{v+1}}, \quad \kappa(z, s) := \frac{v+1}{w(z; v, \bar{c})(s - z)^{v+2}}, \quad (14)$$

straightforwardly satisfy the following second-order partial differential equation

$$\mathcal{L}_z[K(z, s)] = \mathcal{M}_s[\kappa(z, s)]. \quad (15)$$

Now, with these notations and properties at hand, we are looking for an integral representation of the solutions of $\mathcal{L}_z[y(z)] = 0$ which is of the form

$$y(z) = \int_{\Gamma} K(z, s) v(s) ds, \quad (16)$$

being the nucleus $K(z, s)$, defined in Eq. (14), one of the functions satisfying Eq. (15). Having this in mind, applying the operator \mathcal{L}_z (given by Eq. (11)) to Eq. (16) and assuming that its action and integration with respect to s can be interchanged, one obtains

$$\mathcal{L}_z[y(z)] = \int_{\Gamma} \mathcal{L}_z[K(z, s)] v(s) ds = \int_{\Gamma} \mathcal{M}_s[\kappa(z, s)] v(s) ds,$$

which, after use of Lagrange identity (13), become

$$\mathcal{L}_z[y(z)] = \int_{\Gamma} \kappa(z, s) \overline{\mathcal{M}}_s[v(s)] ds - \Delta_{\Gamma}\{f(s)\},$$

where $f(s) := \sigma(s)\kappa(z, s)v(s)$ and $\Delta_\Gamma\{f(s)\}$ stands for the variation of the function $f(s)$ along the contour Γ . So, finally, by choosing the contour Γ in such a way that $\Delta_\Gamma\{f(s)\} = 0$ and taking $v(s) = w_v(z; v, \bar{c})$, so that (as pointed out after Eq. (12)) $\overline{\mathcal{M}}_s[v(s)] = 0$, one has $\mathcal{L}_z[y(z)] = 0$ and the integral representation is obtained. It is Eq. (16) with $K(z, s)$ given by Eq. (14) and $v(s) = w_v(z; v, \bar{c})$.

In this way an alternative proof of [34, Theorem 1] has been given, leading to the definition of what we will understand by hypergeometric-type functions:

Definition 1. A hypergeometric-type function, to be denoted from now on by $y_v(z; \bar{c})$, is any solution of the hypergeometric-type differential equation (6) which admits an integral representation of the form we have just obtained, i.e.,

$$y_v(z; \bar{c}) = \frac{C_v(\bar{c})}{w(z; v, \bar{c})} \int_{\Gamma_v(\bar{c})} \frac{w_v(s; v, \bar{c})}{(s-z)^{v+1}} ds, \quad (17)$$

where $w(z; v, \bar{c})$ is the symmetrization factor of Eq. (6) and $w_v(z; v, \bar{c})$ is the function defined in Eq. (8). Moreover, $C_v(\bar{c})$ stands for a normalizing constant, $v \in \mathbb{C}$ is the parameter defined by Eq. (5) or (10) and the contour $\Gamma_v(\bar{c})$ is such that it satisfies the condition

$$\Delta_{\Gamma_v(\bar{c})} \left\{ \frac{w_{v+1}(s; v, \bar{c})}{(s-z)^{v+2}} \right\} = 0, \quad (18)$$

where, clearly, the variation is taken with respect to the s -variable.

For our purposes here, as fundamental as Eq. (17) is the integral representation for the k th derivative ($k = 1, 2, 3, \dots$) of $y_v(z; \bar{c})$, already given in [34, p. 16], which we are going to construct also in a different way. It can be proved by induction that the k th derivative of any solution of the hypergeometric-type equation (6) satisfies in turn the differential equation

$$\sigma(z)u''(z) + \tau_k(z; v, \bar{c})u'(z) + [\lambda_v + k\tau'_{(k-1)/2}(z; v, \bar{c})]u(z) = 0, \quad (19)$$

which is also of hypergeometric-type. So, we can obtain the corresponding integral representation for their solutions by following the same procedure as above. Defining the new parameter $\mu \in \mathbb{C}$ as a solution of the equation

$$[\lambda_v + k\tau'_{(k-1)/2}(z; v, \bar{c})] + \mu\tau'_k(z; v, \bar{c}) + \frac{\mu(\mu-1)}{2}\sigma'' = 0 \implies \mu = v - k$$

and having in mind that the symmetrization factor of Eq. (19) is related with that of Eq. (6) by $w(z; \mu, \bar{c}) := w_k(z; v, \bar{c})$, this integral representation is

$$u_\mu(z; \bar{c}) \equiv u_{v-k}(z; \bar{c}) = \frac{C_{v-k}}{w_k(z; v, \bar{c})} \int_{\Omega_{v-k}} \frac{w_v(s; v, \bar{c})}{(s-z)^{v-k+1}} ds, \quad (20)$$

where Ω_{v-k} is a contour in the complex plane satisfying

$$\Delta_{\Omega_{v-k}} \left\{ \frac{w_{v+1}(s; v, \bar{c})}{(s-z)^{v+2}} s^k \right\} = 0 \quad (k = 0, 1, 2, \dots). \quad (21)$$

For $k = 0$ this condition coincide with that satisfied by the contour $\Gamma_v(\bar{c})$ of Eq. (17) (see Eq. (18)). So, requiring that $\Gamma_v(\bar{c})$ also satisfies the condition (21), both integral representations (17) and (20) have the same contour for any $k = 0, 1, 2, \dots$. The only thing still missing is to find appropriate constants C_{v-k} in Eq. (20) in such a way that the function $u_{v-k}(z, \bar{c})$ becomes in fact the k th derivative of $y_v(z; \bar{c})$ as given by Eq. (17). It turns out by induction that

$$C_{v-k} := \prod_{j=0}^{k-1} \tau'_{(v+j-1)/2}(z; v, \bar{c}) C_v(\bar{c}), \quad (22)$$

once we have considered that differentiation with respect to z and integration with respect to s can be interchanged, and taking into account the following property satisfied by the symmetrization factor: for $j = 0, 1, 2, \dots$,

$$\frac{d}{dz} \left\{ \frac{w_v(s; v, \bar{c})}{w_{j+1}(z; v, \bar{c}) (s-z)^{v-j+1}} \right\} := \frac{d}{ds} \left\{ \frac{w_v(s; v, \bar{c})}{w_{j+1}(z; v, \bar{c}) (s-z)^{v-j+1}} \right\} + \left\{ \frac{\tau'_{(v+j-1)/2}(z; v, \bar{c}) w_v(s; v, \bar{c})}{w_j(z; v, \bar{c}) (s-z)^{v-j}} \right\}.$$

Summarizing, in this section we have presented an alternative and more natural proof of the following theorem (see [34, Section 3, 4]):

Theorem 1. Assuming that differentiation with respect to z and integration with respect to s can be interchanged, there always exist particular solutions of a hypergeometric-type differential equation (6) which are hypergeometric-type functions in the sense of Definition 1. Moreover, their k th derivatives $y_v^{(k)}(z; \bar{c})$ ($k = 1, 2, 3, \dots$) also have an integral representation of the form

$$y_v^{(k)}(z; \bar{c}) = \frac{C_v^{(k)}(\bar{c})}{w_k(z; v, \bar{c})} \int_{\Gamma_v(\bar{c})} \frac{w_v(s; v, \bar{c})}{(s-z)^{v-k+1}} ds, \quad (23)$$

with a common contour $\Gamma_v(\bar{c})$ satisfying

$$A_{\Gamma_v(\bar{c})} \left\{ \frac{w_{v+1}(s; v, \bar{c})}{(s-z)^{v+2}} s^k \right\} = 0 \quad (k = 0, 1, 2, \dots). \quad (24)$$

Here, $v \in \mathbb{C}$ is the parameter defined by Eq. (5) or (10), $w(z; v, \bar{c})$ is the symmetrization factor of Eq. (6) and $w_v(z; v, \bar{c})$ is the function defined in Eq. (8). Moreover, $C_v^{(k)}(\bar{c}) := C_{v-k}$ are given by Eq. (22) where $C_v(\bar{c})$ stands for a normalizing constant.

Remark 2. When Eq. (5) has the solution $v = n$, being n a positive integer, then Eq. (6) has [10] a n th degree polynomial solution, called *hypergeometric-type polynomial*, which is unique (up to a normalizing constant) if the additional condition

$$n\tau'_{(n-1)/2}(z; n, \bar{c}) \neq m\tau'_{(m-1)/2}(z; n, \bar{c}) \quad (n \neq m),$$

holds. In this case, the contour $\Gamma_v(\bar{c})$ can be chosen as a closed one encircling the point $s = z$ and such that it does not contain the roots of $\sigma(z)$. Then, use of Cauchy's integral formula as applied to the integral representation (23), provides a Rodrigues-type formula for the corresponding polynomial solution and their derivatives, which was also obtained in [34] by completely different means.

4. Construction of recurrence and ladder relationships

We are now in a position to prove our main result, which provides a set of sufficient conditions for the hypergeometric-type functions to satisfy general sum rules of the form Eq. (1) in a constructive way. Indeed we give a method for the explicit computation of the coefficients which fully characterize those relations. To begin with, let us introduce first the concept of hypergeometric-type functions sequence (HTFS) which is the central idea in this procedure, because (see Theorem 2 below) the aforementioned sufficient conditions can be summarized by saying that the functions belong to one of these sequences.

Definition 2. A HTFS $\{y_{v_i}(z; \bar{c}_i)\}_{i=0}^M$ ($M \in \mathbb{N}$), is a family of hypergeometric-type functions (see Definition 1) such that:

- (a) They are solutions of a set of hypergeometric-type differential equations (6) all of them having the same coefficient $\sigma(z)$.
- (b) The differences $v_i - v_j$, ($i, j = 0, 1, \dots, M$), are integer numbers.

(c) The coefficients $\tau(z; v_i, \bar{c}_i)$ in Eq. (6) are related as follows:

$$\tau(z; v_i, \bar{c}_i) - \tau(z; v_j, \bar{c}_j) = E(i, j) \sigma(z) \frac{p'_{i,j}(z)}{p_{i,j}(z)} \quad (i, j = 0, 1, \dots, M), \quad (25)$$

or, equivalently, the symmetrization factors are such that

$$w(z; v_i, \bar{c}_i) = p_{i,j}(z)^{E(i,j)} w(z; v_j, \bar{c}_j) \quad (i, j = 0, 1, \dots, M), \quad (26)$$

where each $p_{i,j}(z)$ can be any divisor of $\sigma(z)$ and the constants $E(i, j)$ are integer numbers.

The latter condition ensures that all members of a HTFS and their derivatives of any order admit an integral representation (23) with a common contour Γ satisfying a condition similar to (24), namely

$$\Delta_\Gamma \left\{ \frac{w_{v_l+1}(s; v_l, \bar{c}_l)}{(s-z)^{v_g+2}} s^k \right\} = 0 \quad (k = 0, 1, 2, \dots), \quad (27)$$

where t ($0 \leq t \leq M$) is an index such that $\forall j = 0, 1, \dots, M$: $E(t, j) \geq 0$ and v_l, v_g are the v_i with lowest and greatest real part, respectively. Moreover, this definition is constructive in the sense that, given a starting hypergeometric-type function $y_{v_0}(z; \bar{c}_0)$, it tells us how to build up any HTFS having this function as one of their elements. For, it is enough to specify the differences $v_0 - v_j$, the integers $E(0, j)$ and the polynomials $p_{0,j}(z)$ for $j = 0, 1, \dots, M$ ($M \in \mathbb{N}$).

Just to illustrate what a HTFS can be, let us consider the hypergeometric-type differential equation satisfied by the Laguerre polynomials $y_n(z; a) := L_n^{(a)}(z)$:

$$zy_n''(z; a) + (a+1-z)y_n'(z; a) + ny_n(z; a) = 0, \quad (28)$$

so that, with the notations of Eq. (6) one has: $\sigma(z) = z$, $\tau(z; v; a) := a+1-z$ and $\lambda_v = n$. The v parameter is then a solution of (10) which in this case reduces to $v = n$, being n the degree of the corresponding Laguerre polynomial. So, the differences $v_i - v_j = n_i - n_j$ always are integer numbers. Moreover, Eq. (25) becomes $\tau(z; v; a) - \tau(z; v; b) = a - b$. In view of this, the sequence of Laguerre polynomials $\{L_n^{(a)}(x)\}_{n \in \mathbb{N}}$ is, of course, an HTFS, but also some more HTFS can be considered. For instance, $\{L_n^{(a+m)}(x)\}_{m \in \mathbb{N}}$ (constructed by keeping the degree n fixed) or $\{L_{n+m}^{(a+m)}(x)\}_{m \in \mathbb{N}}$ (obtained by changing simultaneously the degree n and the parameter a) are also examples of HTFS.

With this definition, our main result can be stated by means of the following

Theorem 2. *Given a HTFS $\{y_{v_i}(z; \bar{c}_i)\}_{i=0}^M$ ($M \in \mathbb{N}$), there always exist N ($3 \leq N \leq M+1$) rational functions $A_{v_i, k_i}(z; \bar{c}_i)$, ($i = 0, \dots, N-1$), such that the sum rule*

$$S_N := \sum_{i=0}^{N-1} A_{v_i, k_i}(z) y_{v_i}^{(k_i)}(z; \bar{c}_i) = 0 \quad (3 \leq N \leq M+1) \quad (29)$$

holds true. Moreover, the $A_{v_i, k_i}(z; \bar{c}_i)$ -coefficients are rational functions of the z -variable; so that Eq. (29) can always be transformed into an equivalent sum rule but with polynomial coefficients.

Proof. To start with, let us define the numbers v_l, v_g and k_l, k_g by

$$\left. \begin{aligned} \Re[v_l] &= \min\{\Re[v_i], i = 0, 1, \dots, N-1\}, \\ \Re[v_g] &= \max\{\Re[v_i], i = 0, 1, \dots, N-1\}, \\ k_l &= \min\{k_i, i = 0, 1, \dots, N-1\}, \\ k_g &= \max\{k_i, i = 0, 1, \dots, N-1\}, \end{aligned} \right\} \quad (30)$$

where $\Re[a]$ stands for the real part of the complex number a . In addition, let r and t ($0 \leq r, t \leq N-1$) be two always existing indexes such that the integers $E(i, j)$ which appears in Eqs. (25)–(26) verify

$$E(i, r) \geq 0, \quad E(t, i) \geq 0, \quad i = 0, 1, \dots, N-1. \quad (31)$$

With these notations, the substitution of the functions $y_{v_i}^{(k_i)}(z; \bar{c}_i)$ in the left-hand side of Eq. (29) by the corresponding integral representation (23) yields

$$\mathbf{S}_N = \int_{\Gamma} \frac{w_{v_l}(z; v_r, \bar{c}_r)}{(s-z)^{v_g-k_l+1}} \mathbf{P}(s; z) ds, \quad (32)$$

where Γ is the common contour satisfying (27). The function $\mathbf{P}(s; z)$, which contains the unknowns $A_{v_i, k_i}(z; \bar{c}_i)$ we are looking for, is given by

$$\mathbf{P}(s; z) := \sum_{i=0}^{N-1} \frac{A_{v_i, k_i}(z; \bar{c}_i) C_{v_i}^{(k_i)}(\bar{c}_i)}{w_{k_i}(z; v_i, \bar{c}_i)} [\sigma(s)]^{v_i-v_l} [p_{i,r}(s)]^{E[i,r]} (s-z)^{v_g-v_i+k_i-k_l}. \quad (33)$$

Recall that $\sigma(s)$ is a polynomial of degree at most two, $p(s)$ is any divisor of $\sigma(s)$ and, as Eqs. (30)–(31) shows, the numbers $v_i - v_l$, $E[i, r]$, $v_g - v_i$, $k_i - k_l$ are always positive integers or zero. So, $\mathbf{P}(s; z)$ is a polynomial in the s -variable of degree

$$\deg_s \{\mathbf{P}(s; z)\} = \max_{i=0, \dots, N-1} \{(v_i - v_l) \deg_s \{\sigma(s)\} + E[i, r] \deg_s \{p_{i,r}(s)\} + v_g - v_i + k_i - k_l\}. \quad (34)$$

Now, in terms of the degree of $\mathbf{P}(s; z)$ as a function of s , two different situations can be considered:

1. $\deg_s \{\mathbf{P}(s; z)\} \leq 1$. In this case, the choice $\mathbf{P}(s; z) = 0$ implies (see Eq. (32)) $\mathbf{S}_N = 0$ and so the searched sum rule (29) holds true. Moreover, this equality $\mathbf{P}(s; z) = 0$ provides a linear system of one ($\deg_s \{\mathbf{P}(s; z)\} = 0$) or two ($\deg_s \{\mathbf{P}(s; z)\} = 1$) equations which are obtained by equating to zero the coefficients of $(s-z)^h$ ($h = 0, 1$). This linear system contains $N \geq 3$ unknowns which are the searched functions $A_{v_i, k_i}(z; \bar{c}_i)$ ($i = 0, \dots, N-1$). So, $N-1$ or $N-2$ can be chosen arbitrarily and the other one or two, respectively, are then obtained from the linear system. Moreover, as a consequence of Eqs. (26), (30) and (31), it turns out that if the equation $\mathbf{P}(s; z) = 0$ is multiplied by $w_{k_g}(z; v_r; \bar{c}_r)$, the coefficients of the linear system become also polynomials in the z -variable. So the unknowns $A_{v_i, k_i}(z; \bar{c}_i)$ ($i = 0, \dots, N-1$) are rational functions of z and the proof of the theorem is completed in this case, since the sum rule (29) can be rewritten but having polynomial coefficients.

2. $\deg_s \{\mathbf{P}(s; z)\} \geq 2$. In this case it may happens that the choice $\mathbf{P}(s; z) = 0$ leads to the trivial solution $A_{v_i, k_i}(z; \bar{c}_i) = 0$ ($i = 0, \dots, N-1$) which is of no interest. However, an alternative method can be devised by mimicking the ideas already used in [20,34,43–46]. For, let us consider an arbitrary function $\mathbf{Q}(s, z)$ except that it is a polynomial in the s -variable of degree two less than the degree of $\mathbf{P}(s, z)$ as given by Eq. (34).

Then it is always possible to choose the unknowns $A_{v_i, k_i}(z; \bar{c}_i)$ ($i = 0, \dots, N-1$) and the function $\mathbf{Q}(s, z)$ in such a way that the integrand in Eq. (32) can be expressed as

$$\frac{w_{v_l}(z; v_r, \bar{c}_r)}{(s-z)^{v_g-k_l+1}} \mathbf{P}(s; z) = \frac{d}{ds} \left\{ \frac{w_{v_l+1}(z; v_r, \bar{c}_r)}{(s-z)^{v_g-k_l}} \mathbf{Q}(s; z) \right\}. \quad (35)$$

If this relation holds, it is then clear that the searched sum rule (29) holds true. This is so because this equality (35) clearly implies $\mathbf{S}_N = 0$ (see Eq. (32)), which is a consequence of condition (27) satisfied by the common contour Γ . But, to see that this choice is always possible the only step required is to perform the derivative on the right side of Eq. (35) to obtain

$$\mathbf{P}(s; z) = \{\tau_{v_l}(s; v_r, \bar{c}_r)(s-z) - (v_g - k_l)\sigma(s)\} \mathbf{Q}(s; z) + (s-z)\sigma(s) \frac{d}{ds} [\mathbf{Q}(s; z)], \quad (36)$$

which shows that $\mathbf{Q}(s; z)$ must be a polynomial in s of degree $\deg_s \{\mathbf{P}(s; z)\} - 2$.

On the other hand, considering the latter equation as a function of s , it provides a linear system which is obtained by expanding both sides in powers of $(s-z)$ and equating the corresponding coefficients. In this system, there are $N + \deg_s \{\mathbf{P}(s; z)\} - 1$ unknowns, N of which are the searched $A_{v_i, k_i}(z; \bar{c}_i)$ functions and the remaining $\deg_s \{\mathbf{P}(s; z)\} - 1$ are the coefficients of $\mathbf{Q}(s; z)$. Moreover, the number of equations is $\deg_s \{\mathbf{P}(s; z)\} + 1$. So, two unknowns are determined in terms of the remaining $N + \deg_s \{\mathbf{P}(s; z)\} - 3$ which can be chosen arbitrarily. This completes the proof of the theorem in the second case and hence of the theorem itself, since for the same reasons explained in the previous case, Eq. (29) can always be transformed into an equivalent sum rule but with polynomial coefficients. \square

As already pointed out, this theorem not only ensures the existence of the algebraic and/or differential properties (29) for any HTFS, but also gives an unified algorithm for the computation of the corresponding coefficients in terms of the coefficients $\sigma(z)$ and $\tau(z; v_i, \bar{c}_i)$. For a specific sum rule, the algorithm consists in solving the linear system which is obtained from Eq. (36) or, in some cases, from the equation $\mathbf{P}(s; z) = 0$.

4.1. Examples

Clearly, Eq. (29) contains very many different sum rules for each HTFS. Here, for illustration, we are going to consider three of the fundamental relationships linking three elements of the HTFS and/or their derivatives of any order, the last two giving rise to ladder operators.

Remark 3. The three- and four-term recurrence formulas explicitly obtained in [20,34,43,44] are particular cases of Eq. (29). Moreover, the hypergeometric-type functions there considered are members of a specific class of HTFS: those for which $p_{i,j}(z) = 1$ in (25) or (26). In such a situation, the coefficients $\tau(z; v_i, \bar{c}_i)$ of (6) are the same for all elements of the HTFS.

4.1.1. Three-term recurrence formula

First of all let us select, among the rich variety of existing HTFS, a family of them. Here (due to its particular interest in several applications) we consider those HTFS defined by choosing the following values for the parameter involved in Eq. (25): $E(i, j) = j - i$ and $p_{i,j}(z) = \sigma(z)$, $\forall i, j = 0, \dots, M$, with $v_i - v_j = i - j$, $\forall i, j = 0, \dots, M$. This means that we are dealing with those HTFS $\{y_{v_i}(z; \bar{c}_i)\}_{i=0}^M$ ($M \in \mathbb{N}$) such that each of its members satisfies the hypergeometric-type differential equation

$$\sigma(z)y'' + \tau(z; v_i; \bar{c}_i)y' + \lambda_{v_i}(\bar{c}_i)y = 0,$$

with

$$\tau(z; v_i; \bar{c}_i) - \tau(z; v_j; \bar{c}_j) = (j - i)\sigma'(z),$$

$$w(z; v_i; \bar{c}_i) = \sigma(z)^{(j-i)} w(z; v_j; \bar{c}_j). \quad (37)$$

Then, the k th derivative of three consecutive elements $y_{v_i+h}(z; \bar{c}_{i+h})$ ($h = -1, 0, 1$) of this family of HTFS satisfies the three-term recurrence relation

$$y_{v_i+1}^{(k)}(z; \bar{c}_{i+1}) = \alpha_{v_i,k}(z)y_{v_i}^{(k)}(z; \bar{c}_i) + \beta_{v_i,k}(z)y_{v_i-1}^{(k)}(z; \bar{c}_{i-1}), \quad (38)$$

where $\alpha_{v_i,k}$ y $\beta_{v_i,k}$ are polynomials (in the z -variable) of degree at most 1 and 2, respectively, given by

$$\alpha_{v_i,k}(z) = \frac{C_{v_i+1}(\bar{c}_{i+1})}{C_{v_i}(\bar{c}_i)} \frac{\tau'_{v_i/2}(z; v_i + 1; \bar{c}_{i+1})}{\tau'_{(v_i+k)/2}(z; v_i + 1; \bar{c}_{i+1})} \frac{\tau_k(z; v_i + 1; \bar{c}_{i+1})}{v_i - k + 1},$$

$$\beta_{v_i,k}(z) = \frac{C_{v_i+1}(\bar{c}_{i+1})}{C_{v_i-1}(\bar{c}_{i-1})} \frac{\tau'_{(v_i+1)/2}(z; v_i + 1; \bar{c}_{i+1})\tau'_{v_i/2}(z; v_i + 1; \bar{c}_{i+1})}{(v_i - k + 1)\tau'_{(v_i+k)/2}(z; v_i + 1; \bar{c}_{i+1})} \sigma(z).$$

Here the notation of Eq. (8) has been used and $C_{v_i+h}(\bar{c}_{i+h})$ ($h = -1, 0, 1$) are the normalizing constants introduced in Eq. (17).

To obtain this recurrence one should apply the method described in the proof of Theorem 2 which, in this case, gives a polynomial $\mathbf{P}(s; z)$ in Eq. (36) of degree 2; so, the function $\mathbf{Q}(s; z)$ is constant with respect to the s -variable. The above relation follows by solving the corresponding system (36) and taking into account Eqs. (22) and (37).

To illustrate the use of the fundamental relation (38), let us consider its expression for Laguerre polynomials $L_n^{(a)}(z)$ satisfying Eq. (28). Taking into account (37), one has that Eq. (38) for this particular system of orthogonal polynomials

yields

$$\frac{d^k}{dz^k}[L_{n+1}^{(a-1)}(z)] = \frac{(n+1)(a+k-z)}{k-n-1} \frac{d^k}{dz^k}[L_n^{(a)}(z)] + \frac{n(n+1)z}{k-n-1} \frac{d^k}{dz^k}[L_{n-1}^{(a+1)}(z)].$$

Having in mind the well known property $(d/dz)(L_n^{(a)}(z)) = L_{n-1}^{(a+1)}$, it turns out that, in fact, the above equation becomes the second-order differential equation satisfied by the k th derivative of $L_{n+1}^{(a-1)}(z)$.

4.1.2. Ladder operators

To derive differential-recurrence relations of the first order, and then to construct ladder operators we consider a HTFS family defined by the choice of the following values for the parameters involved in Eq. (25): $E(i, j) = j - i$ and $p_{i,j}(z) = \sigma(z)$, $\forall i, j = 0, \dots, M$, with $v_i = v$, $\forall i = 0, \dots, M$. This means that we are dealing with a HTFS $\{y_v(z; \bar{c}_i)\}_{i=0}^M$ ($M \in \mathbb{N}$) such that each of its members satisfy the hypergeometric-type differential equation

$$\sigma(z) y'' + \tau(z; v; \bar{c}_i) y' + \lambda_v(\bar{c}_i) y = 0,$$

with

$$\begin{aligned} \tau(z; v; \bar{c}_i) - \tau(z; v; \bar{c}_j) &= (j - i) \sigma'(z), \\ w(z; v; \bar{c}_i) &= \sigma(z)^{(j-i)} w(z; v; \bar{c}_j). \end{aligned} \quad (39)$$

Then, applying the general method described in the proof of Theorem 2 it is possible to obtain the following two differential-recurrence relations:

$$\begin{aligned} H_{v,k}^{(+1)}(z; \bar{c}_i) \frac{d}{dz} \{y_v^{(k)}(z; \bar{c}_i)\} &= F_{v,k}^{(+1)}(z; \bar{c}_i) y_v^{(k)}(z; \bar{c}_i) + G_{v,k}^{(+1)}(z; \bar{c}_i) y_v^{(k)}(z; \bar{c}_{i+1}), \\ H_{v,k}^{(+1)}(z; \bar{c}_i) &= \sigma(z) \{2\sigma' \tau'_{(v-1)/2}(z; \eta; c_i) - \tau(z; \eta; c_i) \sigma''\}, \\ F_{v,k}^{(+1)}(z; \bar{c}_i) &= 2\sigma \tau'_{v-1}(z; v; \bar{c}_i) \tau'_{(v+k-2)/2}(z; v; \bar{c}_i) - \tau_{k-1}(z; v; \bar{c}_i) [2\sigma' \tau'_{v-1}(z; v; \bar{c}_i) - \tau(z; v; \bar{c}_i) \sigma''], \\ G_{v,k}^{(+1)}(z; \bar{c}_i) &= \frac{C_v(\bar{c}_i)}{C_v(\bar{c}_{i+1})} \frac{\tau'_{(v+k-3)/2}(z; v; \bar{c}_i) \tau'_{(v+k-2)/2}(z; v; \bar{c}_i)}{\tau'_{(v-3)/2}(z; v; \bar{c}_i) \tau'_{(v-2)/2}(z; v; \bar{c}_i)} \\ &\quad \times \{\tau_{v-1}(z; \eta; c_i) [2\sigma' \tau'_{(v-1)/2}(z; \eta; c_i) - \tau(z; \eta; c_i) \sigma''] - 2[\tau'_{v-1}(z; \eta; c_i)]^2 \sigma\} \end{aligned}$$

and

$$\begin{aligned} H_{v,k}^{(-1)}(z; \bar{c}_i) \frac{d}{dz} \{y_v^{(k)}(z; \bar{c}_i)\} &= F_{v,k}^{(-1)}(z; \bar{c}_i) y_v^{(k)}(z; \bar{c}_i) + G_{v,k}^{(-1)}(z; \bar{c}_i) y_v^{(k)}(z; \bar{c}_{i-1}), \\ H_{v,k}^{(-1)}(z; \bar{c}_i) &= \tau(z; v; \bar{c}_i) \sigma'' - 2\sigma' \tau'_{v/2}(z; v; \bar{c}_i), \\ F_{v,k}^{(-1)}(z; \bar{c}_i) &= 2\tau'_v(z; v; \bar{c}_i) \tau'_{(v+k-1)/2}(z; v; \bar{c}_i), \\ G_{v,k}^{(-1)}(z; \bar{c}_i) &= -2 \frac{C_v(\bar{c}_i)}{C_v(\bar{c}_{i-1})} \tau'_{v/2}(z; v; \bar{c}_i) \tau'_{(v-1)/2}(z; v; \bar{c}_i). \end{aligned}$$

These two relations allows us to define correspondingly the two following operators:

$$T_{v,k}^{(\pm 1)}(z; \bar{c}_i) := H_{v,k}^{(\pm 1)}(z; \bar{c}_i) \frac{d}{dz} - F_{v,k}^{(\pm 1)}(z; \bar{c}_i)$$

which are such that

$$T_{v,k}^{(\pm 1)}(z; \bar{c}_i) \{y_v^{(k)}(z; \bar{c}_i)\} = G_{v,k}^{(\pm 1)}(z) y_v^{(k)}(z; \bar{c}_{i \pm 1}).$$

So, they are ladder operators in the sense that they shift by ± 1 the parameters \bar{c}_i . In particular, for the Laguerre polynomials, these two ladder operators get simplified as

$$\left\{ z \frac{d}{dz} + (a + k) \right\} [L_n^{(a)}(z)]^{(k)} = (a + n) [L_n^{(a-1)}(z)]^{(k)},$$

$$\left\{ 1 - \frac{d}{dz} \right\} [L_n^{(a)}(z)]^{(k)} = [L_n^{(a+1)}(z)]^{(k)}, \quad (40)$$

respectively.

5. Generalized hypergeometric-type functions

As pointed out in the introduction, there always exists (see [34,45]) a change of variable (in general not unique) which transforms any generalized hypergeometric-type differential equation (2) into an equation of hypergeometric-type (6). So, all the recurrence and differential-recurrence relations considered in the previous section can be extended to the solutions of the generalized equation (2).

For this extension to be carried out the main idea is that of *generalized hypergeometric-type functions sequence* (GHTFS). Before defining this concept, let us summarize the way of obtaining the aforementioned change of variable.

Proposition 1. *With the notation of Eq. (2), let $\eta \in \mathbb{C}$ be a constant such that the function*

$$r(z) := \sqrt{\left(\frac{Q'(z) - p(z)}{2} \right)^2 - q(z) - \eta Q(z)}$$

is a polynomial of degree at most one. Moreover, let $\pi_{\pm}(z)$ be also a polynomial defined by

$$\pi_{\pm}(z) := \frac{Q'(z) - p(z)}{2} \pm r(z).$$

Then, the change of variable

$$u(z) = y(z) \exp \left\{ \int \frac{\pi_{\pm}(z)}{Q(z)} dz \right\} \quad (41)$$

transforms Eq. (2) into a hypergeometric-type differential equation

$$\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0$$

with

$$\sigma(z) := Q(z), \quad \tau(z) := p(z) + \pi_{\pm}(z), \quad \lambda := \pi'_{\pm}(z) - \eta. \quad (42)$$

Proof. See [34, pp. 1–3]. \square

Definition 3. A GHTFS $\{U_{v_i}(z; \bar{c}_i)\}_{i=0}^M$ ($M \in \mathbb{N}$) is a family of solutions of Eq. (2) such that each of its elements is linked with an element of a HTFS by means of (41), so that for $i = 0, \dots, M$

$$U_{v_i}(z; \bar{c}_i) = y_{v_i}(z; \bar{c}_i) \exp \left\{ \int \frac{\pi_{\pm}(z)}{Q(z)} dz \right\},$$

where $\{y_{v_i}(z; \bar{c}_i)\}_{i=0}^M$ is a HTFS as introduced in Definition 2.

The extension of the recurrence and differential-recurrence relations from HTFS to GHTFS is now a simple matter which just requires the explicit determination of the change (41) as indicated in Proposition 1.

6. Radial wave functions of the N -dimensional hydrogen atom

As it is well known the radial wave functions of the hydrogen atom in $D \geq 3$ dimensions can be written in terms of Laguerre polynomials as [33,42]:

$$R_{m,L}^{(D)}(r) = \left[\frac{(m-L-1)!}{(m+L+D-3)!} \right]^{1/2} \left[\frac{2^D}{(2m+D-3)^{(D+1)/2}} \right] \left[\frac{4r}{(2m+D-3)} \right]^L \\ \times \exp \left\{ -\frac{2r}{(2m+D-3)} \right\} L_{m-L-1}^{(2L+D-2)} \left(\frac{4r}{(2m+D-3)} \right).$$

This means that $R_{m,L}^{(D)}(r)$ is a solution of a generalized hypergeometric-type differential equation such that, by means of the change of variable described in Proposition 1, can be transformed into an equation of hypergeometric-type (in this case the differential equation satisfied by Laguerre polynomials). So, the sum rules satisfied by any HTFS made up of these polynomials provide recurrence and differential-recurrence relations for the corresponding hydrogen-like wave functions.

For instance, assume we want to construct ladder operators for $R_{m,L}^{(D)}(r)$ shifting the dimension D by ± 1 , but leaving the remaining quantum numbers unchanged. In this case we are dealing with the GHTFS (see Definition 3) $\{R_{m,L}^{(D+h)}(r)\}_{h=D-3,D-2,\dots}$. Then to change the dimension D by ± 1 , the corresponding ladder operator shifting by ± 1 the a -parameter of Laguerre polynomials $L_n^{(a)}(z)$ are to be used. These operators are, precisely, those already computed in Section 4.2 (see Eq. (40)). Using them we obtain:

$$\left\{ 2 + (2m+D-2) \left[\frac{L}{r} - \frac{d}{dr} \right] \right\} R_{m,L}^{(D)} \left(\frac{2m+D-3}{2m+D-2} r \right) \\ = 2(m+L+D-2)^{1/2} \frac{(2m+D-2)^{D/2+1}}{(2m+D-3)^{(D+1)/2}} R_{m,L}^{(D+1)}(r), \quad D \geq 3$$

and,

$$\left\{ (2m+D-4) \left[(L+D-2) + r \frac{d}{dr} \right] + 2r \right\} R_{m,L}^{(D)} \left(\frac{2m+D-3}{2m+D-4} r \right) \\ = 2(m+L+D-3)^{1/2} \frac{(2m+D-4)^{D/2+1}}{(2m+D-3)^{(D+1)/2}} R_{m,L}^{(D-1)}(r), \quad D \geq 4.$$

Summing up, we have a constructive procedure to explicitly obtain recurrence and differential-recurrence relations among not only hypergeometric-type functions but also generalized hypergeometric-type functions, which satisfy second-order differential equations with parameter-dependent polynomials coefficients. This procedure allows us to derive all the corresponding published relations for the classical orthogonal polynomials [3,19,20,22,28,29,34,43,44] as well as for the radial wave functions of the D -dimensional physical prototypes [4,9,11,32,45,46], including recurrence and ladder relations which link functions with different dimensionality.

Acknowledgments

The authors wish to acknowledge partial financial support from Ministerio de Ciencia y Tecnología of Spain under the grants MTM2005-08648-C02-01 (A.Z.) and FIS2005-00973 (R.J.Y., J.S.D.), and European Research Network “Network on Constructive Complex Approximation (NeCCA)”, INTAS 03-51-6637. We belong to the PAI group FQM-0207 of Junta de Andalucía.

References

- [1] M. Abramowitz, I. Stegun, Handbook of Mathematical Functions, Dover, New York, 1965.
- [2] V. Aldaya, J. Bisquert, J. Navarro-Salas, The quantum relativistic harmonic oscillator: generalized Hermite polynomials, *Phys. Lett. A* 156 (1991) 381.
- [3] W.A. Al-Salam, Characterization theorems for orthogonal polynomials, in: P. Nevai (Ed.), *Orthogonal Polynomials: Theory and Practice*, Kluwer Academic Publishers, New York, 1990.
- [4] R. Álvarez-Nodarse, J.L. Cardoso, N.R. Quintero, On recurrence relations for radial wave functions for the N -th dimensional oscillators and hydrogen like atoms: analytical and numerical study ETNA, *Proceedings of the IWOP04*, 2005.
- [5] G.E. Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.
- [6] L.C. Andrews, *Special Functions for Engineers and Applied Mathematics*, Macmillan Publishing Company, New York, 1985.
- [7] V.G. Bagrov, D.M. Gitman, *Exact Solutions of Relativistic Wave Equations*, Kluwer Academic Publishers, Dordrecht, 1990.
- [8] N. Barik, K -th differences of functions satisfying three term recurrence relations (I) and (II), *Pure Math. Manuscript* 3 (1984) 11–21 4 (1984) 47.
- [9] P.K. Bera, S. Battacharyya, U. Das, B. Talukdar, Recurrence relations for N -dimensional radial wave functions, *Phys. Rev. A* 48 (1993) 4764.
- [10] S. Bochner, Über Sturm–Liouvillesche polynomsysteme, *Math. Z.* 89 (1929) 730.
- [11] J.L. Cardoso, R. Álvarez-Nodarse, Recurrence relations for radial wavefunctions for the N th-dimensional oscillators and hydrogenlike atoms, *J. Phys. A: Math. Gen.* 36 (2003) 2055.
- [12] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon & Breach, New York, 1978.
- [13] G. Dattoli, A. Dipace, Remarks of the derivation of sum rules of special functions, *Nuovo Cimento* 90B (1985) 85.
- [14] G. Dattoli, A. Dipace, A. Torre, About the sum rules of special functions, *Nuovo Cimento* 90B (1985) 85.
- [15] G. Dattoli, V. Loreto, C. Mari, M. Richetta, A. Torre, Biunitary transformations and ordinary differential equations I, *Nuovo Cimento* 106B (1991) 1357.
- [16] G. Dattoli, V. Loreto, C. Mari, M. Richetta, A. Torre, Biunitary transformations and ordinary differential equations II, *Nuovo Cimento* 106B (1991) 1375.
- [17] G. Dattoli, V. Loreto, C. Mari, M. Richetta, A. Torre, Biunitary transformations and ordinary differential equations III, *Nuovo Cimento* 106B (1991) 1391.
- [18] J.S. Dehesa, F. Domínguez-Adame, E.R. Arriola, A. Zarzo, Hydrogen atom and orthogonal polynomials, in: C. Brezinski, L. Gori, A. Ronveaux (Eds.), *Proceedings of the Third International Symposium on Orthogonal Polynomials and Their Applications*, IMACS Annals of Computational Applied Mathematics, vol. 9, J.C. Baltzer AG, Basel, 1991, p. 223.
- [19] J.S. Dehesa, R.J. Yáñez, M. Pérez-Victoria, A. Sarsa, Non linear characterizations for functions of hypergeometric-type and their derivatives of any order, *J. Math. Anal. Appl.* 184 (1994) 35.
- [20] J.S. Dehesa, R.J. Yáñez, A. Zarzo, J.A. Aguilar, New linear relationships of hypergeometric-type functions with applications to orthogonal polynomials, *Rend. Mat. (Ser. VII)* 13 (Roma) (1993) 661.
- [21] P.A.M. Dirac, *The Principles of Quantum Mechanics*, Oxford U.P., Oxford, 1958 (see also *Proc. R. Soc. London Ser. A* 109 (1925) 642).
- [22] A. Erdélyi, Bateman Manuscript Project, *Higher Transcendental Functions*, A. Erdélyi (Ed.), Mc-Graw Hill, New York, 1955.
- [23] E.R. Hansen, Sums of functions satisfying recursion relations, *Amer. Math. Monthly* 88 (1981) 676.
- [24] E.L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.
- [25] L. Infeld, T.E. Hull, The factorization method, *Rev. Mod. Phys.* 23 (1951) 21.
- [26] T. Inoui, Unified theory of recurrence formulas, *Prog. Theoret. Phys.* 3 (1948) 169 244.
- [27] O.L. de Lange, *Operational Methods in Quantum Mechanics*, Clarendon Press, Oxford, 1991.
- [28] F. Marcellán, A. Branquinho, J. Petronhilo, Classical orthogonal polynomials: a functional approach, *Acta Appl. Math.* 34 (1994) 283.
- [29] P. Martínez, R.J. Yáñez, A. Zarzo, Properties of the relativistic Hermite polynomials from the differential equation that they satisfy, in: D. Cocolicchio, G. Dattoli, H. Srivastava (Eds.), *Proceedings of the Workshop “Advanced Special Functions and Applications” Melfi, Italy, May 1999*, Aracne Editrice, Rome. *Proc. Melfi Sch. Adv. Top. Math. Phys.* 1 (2000) 127.
- [30] W. Miller, Jr., *Symmetry and Separation of Variables*, Encyclopedia of Mathematics and Its Applications, vol. 4, Addison-Wesley, Reading, MA, 1977.
- [31] J. Morales, G. Areaga, J.J. Peña, J.J. López-Bonilla, Alternative approach to the factorization method, *Int. J. Quantum Chem.* 26 (1992) 171.
- [32] J. Morales, J.J. Peña, J.J. López-Bonilla, Algebraic approach to matrix elements: recurrence relations and closed formulas for hydrogen-like wavefunctions, *Phys. Rev. A* 45 (1992) 4259.
- [33] M.M. Nieto, Hydrogen atom and relativistic pi-mesic atom in N -space dimensions, *Am. J. Phys.* 47 (1979) 1067.
- [34] A.F. Nikiforov, V.B. Uvarov, *Special Functions of Mathematical Physics*, Birkhauser, Basel, 1988.
- [35] F.J.W. Olver, *Asymptotics and Special Functions*, Academic Press, New York, London, 1974.
- [36] J. Peetre, Correspondence principle for the quantized annulus: Romanovski polynomials and Morse potentials, *J. Funct. Anal.* 117 (1993) 377.
- [37] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, *Integrals and Series*, 3 Volumes, Gordon & Breach, New York, 1986.
- [38] W. Schultz-Piszachich, Eine klasse isotroper turbulenzfelder mit permanenzeigenschaft und existenten tripelkorrelationen, *Z. Angew. Math. Mech.* 56 (1976) 303.
- [39] H.M. Srivastava, Some orthogonal polynomials representing the energy spectral functions for a family of isotropic turbulence fields, *Z. Angew. Math. Mech.* 64 (1984) 255.
- [40] N.M. Temme, *Special Functions: An Introduction to the Classical Functions of Mathematical Physics*, Wiley, New York, 1996.
- [41] C. Truesdell, *An Essay Toward a Unified Theory of Special Functions*, Annals of Mathematics Studies, No. 18, Princeton University Press, Princeton, NJ, 1948.

- [42] R.J. Yáñez, W. van Assche, J.S. Dehesa, Position and momentum information entropies of the D -dimensional harmonic oscillator and hydrogen atom, *Phys. Rev. A* 50 (1994) 489.
- [43] R.J. Yáñez, J.S. Dehesa, A.F. Nikiforov, The three-term recurrence relation and the differentiation formulas for hypergeometric-type functions, *J. Math. Anal. Appl.* 188 (1994) 855.
- [44] R.J. Yáñez, J.S. Dehesa, A. Zarzo, Four-term recurrence relations of hypergeometric-type polynomials, *Nuovo Cimento* 109B (1994) 725.
- [45] A. Zarzo, Hypergeometric-type differential equations, Ph.D. Thesis, University of Granada, Granada, Spain, September 1995 (in Spanish).
- [46] A. Zarzo, J.S. Dehesa, J. Torres, On a new set of polynomials representing the wave functions of the quantum relativistic harmonic oscillator, *Ann. Numer. Math.* 2 (1995) 439.